An analysis of visual detection by temporal probability summation

U. Mortensen*

Fachbereich Psychologie und Sportwissenschaften, Institut III, Westfälische Wilhelms-Universität, Fliednerstr. 21, D-48149 Munster, Germany

Abstract

The probability of detection by temporal probability summation (TPS) is derived assuming a non-white Gaussian noise process, characterised by a finite second spectral moment λ_2; the assumption of stationary noise is discussed with respect to neurophysiological findings suggesting multiplicative noise for individual neurons. The usual white-noise assumption is shown to imply inconsistent detection models. d'-estimates for 2AFC-experiments do not depend on the value of λ_2 and do not differ from those derived from the standard SDT-assumptions, in contrast to d'-estimates for yes–no-experiments, which depend on the probability of false alarms. Estimates of properties of a visual channel depend on whether detection is by TPS or temporal peak detection (TPD); it is argued that TPD is not a special case of TPS, and that subjects chose a TPS- or a TPD-strategy depending on experimental conditions.
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1. Introduction

Since the work of Sachs, Nachmias, and Robson (1971), the notion of probability summation[1] has become ubiquitous in visual psychophysics. In particular, Quick's (1974) model became a fundamental component of a variety of detection models in which probability summation among independent channels is assumed, e.g. Graham (1977, 1989), Wilson and Bergen (1979), Robson and Graham (1981), Pelli (1985, 1987). As it seems, the notion of probability summation and Quick's formula have become a synonym for each other; see, e.g. Zele and Vingrys (2000) and Manahilov, Calvert, and Simpson (2003). For example, in the latter it is explicitly stated, on page 1858, that "According to the probability summation concept, the threshold contrast can be defined by the Quick formula . . .". Watson and Nachmias (1977) and Watson (1979) presented an adaptation of Quick's model for temporal probability summation (TPS) which acquired the status of an almost canonical or standard model for detection processes; see, e.g. Watson (1982), Georgeson (1987),

Blommaert and Roufs (1987), Schofield and Georgeson (2000). Seen from a mathematical angle, Quick's and Watson's (1979) model are very convenient; whether this property is enough to justify their use without caution concerning its validity remains to be discussed (cf. Section 2.2).

According to the notion of detection by TPS the stimulus is detected when the maximum of the activity exceeds some threshold value within some specified time interval J. So the task is to derive the distribution function of the maximum of activity during the interval J, given an arbitrary stimulus. Since the activity has to reach a threshold within the interval J the problem of deriving the distribution of the maximum is equivalent to that of deriving a first-passage-time distribution. The key assumptions are (1) the activity in the detecting channel is noisy, and (2) the stimulus may be detected, at least in principle, at any time during J. It follows that the probability of detection depends on the complete temporal course of the neural activity during a trial.

The assumption of detection by TPS may be contrasted with that of detection by temporal peak detection (TPD), where assumption (2) is replaced by the assumption that the probability of detection only depends on the maximum value of the mean activity generated by the stimulus. If the TPD-assumption holds, it can be shown that the reciprocal of the threshold contrast of the stimulus is proportional to

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*Fax: +49 251 83 39105.

E-mail address: mortens@psy.uni-muenster.de.

[1]However, the expression "probability summation" is misleading in many respects and is not standard in the literature on statistics or stochastic processes; it appears to be idiosyncratic to visual psychophysics. It will be used here because it is well established in this area of research.

the maximum of the unit response to the stimulus, i.e. to the maximum of the response to the stimulus with contrast or amplitude equal to 1, provided the detecting visual channel can be approximated by a linear system for contrasts within threshold range. If detection by TPD can be assumed, the psychometric function can be defined with respect to a distribution function of some random variable; however, usually no particular distribution function has to be postulated, all that has to be assumed is that the probability of detection equals a certain constant if the maximum of the mean response equals some other constant. It is then possible to extract from the data properties of the detecting channel directly; in particular, the temporal course of the mean response to a stimulus may be recovered, provided the detecting channel can be approximated by a linear system (Roufs & Blommaert, 1981). The TPD-assumption has been employed in a large body of experimental work, e.g. that of Kelly (1971a, 1971b), Kulikowski and King-Smith (1973), Roufs (1972a, 1972b), Graham (1977, 1989), Wilson and Bergen (1979) and Bonneh and Sagi (1998), to name just a few. More recent models of population codes may also be seen as making use of the TPD-assumption: for instance, Deneve, Latham, and Pouget (1999) proposed a model for the reading of population codes where the activity of spatial frequency and orientation channels develops over time towards certain mean values which are then postulated to be relevant for the detection or identification task; see also Pouget, Dayan, and Zemel (2003). The adequacy of the TPD-assumption may be tested by predicting the results from one experiment from the results of another experiment; this has been done e.g. by Roufs and Blommaert (1981), Meinhardt and Mortensen (1998). If, on the other hand, detection by TPS is assumed, the psychometric function is also defined with respect to a distribution function of a random variable, but this time the random variable represents the maximum of the channel's activity during some time interval. The distribution function of this random variable depends (i) on the complete course of the mean activity of the channel, and (ii) on the structure of the noise process. The sensitivity, i.e. the reciprocal of the threshold contrast, does no longer relate in a simple way to the mean value function of the activity. In contrast to the TPD-case, explicit assumptions about the noise are necessary in order to derive properties of the detecting channel.

The main aim of this paper is to present some expressions for the distribution function of the maximum of the activity. The expressions are derived making use of results from the theory of extremes. In particular, the following points will be discussed:

1. A frequently made assumption in psychophysics is that the noise is "white". It will be shown that postulating this type of noise implies an inconsistent model of detection; the interpretation of data in terms of TPS-models based on the assumption of white noise may be flawed.

2. A distribution function—and the corresponding psychometric function—for the maximum of the activity during a given time interval will be presented that is not based on the assumption of white noise, and some of its properties, like the relation between d' and the corresponding probability of detection, will be discussed.

3. It will be argued that TPD is not a special case of TPS. It is suggested that TPS and TPD are different modes of detection and the subject decides which mode to choose depending on the experimental conditions. An illustration of the application of the TPS-model to detection data that are compatible with the hypothesis of detection by TPD and not by TPS will be given in a separate paper (Mortensen, 2007).

The models presented in this paper refer to psychophysical, not neurophysiological experiments. If detection is assumed to be TPD, usually no detailed assumptions concerning the structure of the noise are usually made; however, the claim that the assumption of TPD is equivalent to assuming that the detecting channels are not noisy at all (Watson, 1982) will be shown not to be valid. On the other hand, the assumption of detection by TPS requires more explicit assumptions concerning the noise in the detecting channel. In a psychophysical context, it is usually postulated that the noise is a stationary Gaussian process. As the following brief survey of the literature on noise in neural systems shows, this necessarily simplifying assumption may be hard to swallow for any researcher who focusses on neurophysiological questions. Some of the results presented in the following may, however, provide some justification of the assumptions made in this paper.

The activity of a neuron may be represented in terms of its firing rate or in terms of the interspike times. The firing rate is found—for given stimulus conditions—to vary about some mean value from one experimental trial to the next. This variability may be referred to as "noise". The activity is extended in time, and so the mean will be a function of time, the mean-value function. Similarly, the variance of the activity may be a function of time. If the variance is constant over time, the noise is often called additive, and if the standard deviation (SD), i.e. the root of the variance, is proportional to the mean, the noise is called multiplicative (Dayan & Abbott, 2001); earlier investigations concerned with multiplicative noise were presented e.g. by Lilliywhite (1981), Teich (1981) and Teich, Prucnal, Vannucci, Breton, and McGill (1982). There are alternative, more general ways of defining additive and multiplicative noise, but the definition given here will be sufficient for the moment.

The assumption of additive noise is common in psychophysics; however, the question is whether this assumption is justified. Tolhurst, Movshon, and Thompson (1981) found that the variance and the mean number of action potentials of simple and complex cells of the cat's visual cortex in response to a grating were positively

related; the relation could approximately be described by a power function with exponent between 1.0 and 1.4. They found further that below a certain stimulus contrast the activity of the neurons differed only little from the level of spontaneous activity. The relation between mean and variance function was also investigated by, e.g. Levine (1992) and Levine, Clelland, Mukherjee, and Kaplan (1996). These authors considered log-variance versus log-rate plots, where rate means spike rate (recorded extracellularly in the optical tract of cats), and found that the variance increases faster for LGN-cells than for retinal ganglion cells. According to Levine et al., the advantage of a higher variance for stronger stimuli may be seen in the fact that the visual system is an interlocked system of (sub-)populations of neurons whose task is to "settle into a solution" (Rumelhart & McClelland, 1986), i.e. to find, if possible, the absolute minimum in a multidimensional solution space. Noise reduces the probability that the system settles in a local minimum. If the SD is proportional to the mean, the coefficient of variation is constant, or the signal-to-noise ratio is about constant. This would imply that there is always the right amount of noise: if there is too little noise the system may settle in wrong solutions too often, and if there is too much noise the system cannot settle at any solution. Since the variability of the neuronal activity is not contained in the perception of the world one may argue that the variability is either somehow removed in later processing stages or is an integral part of the visual processing that implies perception (Levine et al., 1996).

Apart from photon noise, which may have to be taken explicitly into account for weak stimuli (Burgess, 1999), the noise results from the fact that the neural channels are noisy devices. Several types of noise may be distinguished: thermal noise, i.e. voltage fluctuations that may be traced to thermal ionic movements in the cell membrane, shot noise, i.e. currents generated by ion transits through the membrane, further flicker noise, i.e. voltage noise with a level that is greater at low frequencies than expected by thermal and shot noise, and conductance fluctuations; the membrane conductance will fluctuate as the number of open ion channels varies. Reviews of earlier work together with an explicit discussion of corresponding models were given by Holden (1976) and De Felice (1981).

A more recent discussion of conductance noise may be found in White, Rubinstein, and Kay (2000), who consider what they call channel noise, which is the probabilistic gating of voltage-dependent ion channels. Brunel, Chance, Fourcaud, and Abbott (2001) argue that the primary source of noise is due to background synaptic activity. Neurons are in principal nonlinear systems, and this noisy activity has a considerable impact on the response dynamic of nonlinear systems. Destexhe, Rudolph, and Paré (2003) point out that neocortical neurons, in intact networks, are under intense synaptic bombardment and are in a high-conductance state. Such states alter the integrative properties of cortical neurons. These neurons have an extremely dense connectivity: each neuron has between 5000 and

60 000 synaptic contacts. Synaptic noise, resulting from the discharge activity of a large number of presynaptic neurons, causes an irregular threshold dynamics of the membrane potential of neurons *in vivo*, in contrast to *in vitro* situations. The power (square of amplitude) spectrum is broadband, corresponding to coloured noise; it is approximately proportional to $1/f^k$, where $k = 2.6$ and f represents frequency (Destexhe et al., Box 1). One finds an enhanced responsiveness to low-amplitude inputs, known as *stochastic resonance*, which is found in nonlinear systems, yielding peak values of the signal-to-noise ratio and thus facilitates the detection of stimuli. Stacey and Durand (2001) have shown that stochastic resonance may also result from neuronal endogenous noise. Demb, Sterling, and Freed (2004) investigated synaptic noise at retinal ganglion cells and measured the intracellular response, i.e. (i) the subthreshold membrane potential V_m, and (ii) the spike times. For retinal ganglion cells, the spike responses (i.e. the number of spikes per second or the spike rate), as well as the amplitude of V_m increased nonlinearly with the stimulus level, whereas the SD remained constant, i.e. the noise was independent of contrast. So it appears that the retinal output can be characterised as a signal plus additive noise, i.e. noise that is independent of the signal, so that "early" vision improves proportional to the input level (Sun, Rüttiger, & Lee, 2004). In the primary visual cortex, however, the noise—expressed in terms of the SD of the activity—increases with the input level.

Wolfart, Debay, Le Masson, Destexhe, and Bal (2005) investigated thalamic relay neurons and found that the duality of burst and single-spike modes, found in *in vitro* experiments, is strongly modified by synaptic background noise, which, together with conductance noise, also influences the input–output transfer function of these neurons. For thalamic neurons the authors find multiplicative scaling by synaptic noise, meaning gain reduction and increased responsiveness for small inputs. Thalamocortical cell bursting is found to be part of an overall synchronised activity during sleep, while during activity there is a mixing of single spike and bursting mode; together with the background synaptic activity an approximately linear response over the input range is generated. The cortex may exert control over the relay function by regulating the background activity through instantaneous adjusting of gain and bursting activity, which may reflect the operation of attentional mechanisms.

An important question refers to the way stimuli are encoded. According to one assumption stimuli are encoded by the spike rate; according to Stein, Gossen, and Jones (2005), this hypothesis was first suggested by Adrian and Zotterman (1926). One may briefly speak of the "rate code". Stein et al. argued, however, that the exact timing of spikes in a population of neurons may carry the information about the stimuli; this is the "temporal coding" hypothesis. Reich, Victor, Knight, and Ozaki (1997) found that neuronal spike trains can exhibit high,

stimulus-dependent temporal precision even while the trial-to-trial response variability remains independent of the stimulus, which applies to retinal ganglion cells as well as to LGN cells. According to these authors the size and the variability of the neuronal responses cannot be analysed in a straightforward manner—several measures of variability have to be considered. Reich et al. consider three measures: (i) the SD of the fundamental Fourier components for each response cycle, (ii) the SD of the spike count in the same cycle, and (iii) the variability of spike timing, and found no obvious relationship between these measures. In any case, the variability is always generated by background noise, resulting from the input of excitatory and inhibitory inputs from many other cells affecting the membrane potential of the cell, implying fluctuations in the firing threshold. Reich, Mechler, and Victor (2001) provide some detailed evidence that temporal coding allows to distinguish small contrast differences, whereas rate codes are employed for gross contrast discriminations; the authors argue—on the basis of information theoretic considerations—that taking into account the temporal structure of neuronal responses could mean an enlargement of the dynamic range for contrast encoding. Steinmetz et al. (2000) showed that cortical neurons become increasingly synchronised when the animal was paying attention, in particular in difficult sensory discrimination tasks; this may be taken as evidence for the importance of spike timing, and that information transmission is by a population of neurons, rather than by a single neuron.

One problem with a rate code is that it can detect changes in the stimulus situation only after a certain number of spikes has been sampled. For this reason, signal encoding by rates is relatively slow in particular when the stimulus is meant to be signalled by single neurons. On the other hand, signal processing in the cortex is fast (Thorpe, Fize, & Marlot, 1996). This speed can be accounted for making the plausible assumption that the stimulus is processed by populations of neurons, where the stimulus is encoded by the population rate. This is the average number of spikes in the population in a time interval. As Silberberg, Bethge, Markram, Pawelzik, and Tsodyks (2004) point out the population rate can be estimated on a time scale that is smaller than the interspike intervals of the individual neurons. The larger the encoding ensemble of neurons the higher the temporal precision. Silberberg et al. considered in particular the instantaneous distribution of input currents into the neurons of a population and found that the synaptic inputs can be subdivided into two components: one is given by ensemble average, the other by the deviations from the average; this is the population variance. The response of an ensemble is then given by an instantaneous population rate, which can be estimated by the number of spikes generated by the population in a small interval, divided by the number of neurons. The population rate is influenced by the amplitudes of both input components. Silberberg et al. show that correlated input currents cannot transmit rapidly changing signals, whereas the encoding of such stimuli in the variance can, independent of the membrane time constants of the individual neuron in the population.

There are good reasons to assume stimulus encoding in population rates. It is found that the firing of cortical neurons is highly irregular when measured *in vivo*, as compared to measurements *in vitro*; in fact, the spike trains "have more in common with a Geiger counter than of a clock" (Stevens & Zador, 1998), a finding already discussed by Softky and Koch (1993). Stevens et al. argued and provided corresponding evidence that the high variability is due to large and rapid fluctuations of the synaptic drive which could result from nearly synchronous firing of subpopulations of afferents. While the findings of Stevens et al. do not yet exclude the possibility of temporal coding, they point to the activity of subpopulations that may play a role in the detection process even if this activity is not due to the stimulus. So one may agree with Harris (2005) who pointed out that a stimulus most likely modulates ongoing activity in cell assemblies instead of controlling firing patterns in individual neurons deterministically. Harris defined four signatures of the activity in cell assemblies: (1) spike trains show temporal structure that is not present in the stimulus, (2) spiking is not controlled by sensory input, (3) spike times are coordinated such as to reveal the organisation of an assembly, and (4) patterns of activity in an assembly should correlate with ongoing "internal cognitive process" (ICPs). The stimulus will most likely activate a chain of assemblies, called *phase sequence* by Harris, so if one starts with the reasonable assumption that detection is the result of a sufficient activation of some cell assembly this activity will represent a mixture of activities from other assemblies.

The results on noise in the visual system yield a complex picture, compared to which the assumptions concerning the noise usually adopted by psychophysicists—e.g. stationary noise—may appear to be rather simplistic. On the other hand, one may follow the argument that the detection of a stimulus is the result of the modulation of activity of certain cell assemblies, which could well mean that the specificities of the different kinds of noise observed in individual neurons need not be taken into account when it comes to relate stimuli and responses in psychophysical experiments. Further, at present any attempt to derive the structure of the noise at the site of detection from the various types of noise at earlier stages seems to be unrealistic. In a way, the relation between psychophysics and neurophysiology is analogous to the relation between thermodynamics and statistical mechanics. To make use of a term from philosophy of science one may say that thermodynamics can be *reduced* to statistical mechanics. Unfortunately, an analogous reduction of psychophysics to neurophysiology in the sense of Nagel (1961), who provided a conceptual framework to "reconstruct" the reduction of thermodynamics to statistical mechanics, has not yet been established. Such a reduction would provide a deduction of the noise structure

at the site of detection from the various forms of noise at earlier stages. One difficulty with such a project is that at present there are too many open questions in psychophysics as well as in neurophysiology, another difficulty is that the notion of reduction itself is still under discussion (Bickle, 1998). The assumptions concerning the noise adopted in this paper are thus understood as representing first approximations to complicated processes.

Overview: Section 2 contains the main results. In Section 2.1 assumptions and definitions will be given; there, the possibility that TPD is a special case of detection by TPS is also discussed in general terms. In Section 2.2 psychometric functions will be derived on the basis of the commonly made assumption of δ-correlated noise; Watson's (1979) psychometric function may be seen as a special case. The assumption of δ-correlated noise is shown to imply an inconsistency, resulting from the fact that this type of noise does not exist in reality; while the assumption of δ-correlated noise may provide useful simplifications in expressions resulting for the case of non-δ-correlated noise, it may lead to wrong conclusions about the detection processes involved when adopted for a start. In Section 2.3, a psychometric function for the case of detection by TPS in correlated Gaussian noise is derived, making use of results from the theory of extremes of dependent variables; in this section, also the notion of *effect of the noise* is discussed. In Section 2.4, an alternative result is presented, derived by Ditlivsen (1971) on the basis of assumptions that are less restrictive than those underlying the extreme-values approach. Ditlivsen's result may thus serve to check the validity of the extreme-values result; however, both results yield practically identical results when applied to detection data, which justifies the mathematically simpler extreme-values approximation. In Section 2.5, the relation between d' and a parameter defining a noise characteristic is discussed, and eventually in Section 2.6 questions of parameter identification are considered. In Section 3, a brief summary and discussion will be given. The results will be discussed in greater detail in the accompanying paper, since then the results of an evaluation of the TPS-hypothesis with respect to some empirical data will be available.

2. Detection by TPS and TPD

A stimulus is detected by some neuronal subsystem of the visual system, which may be subdivided into further subsystems or "channels". In this paper the focus will be on detection by a single channel, mainly for reasons of simplicity, but also because the data that will be discussed appear to justify this assumption.

2.1. Definitions and assumptions

All considerations in this paper are based on the assumption that sensory activity can be represented by some continuous-parameter stochastic process[2] $\{X_t, t \in J\}$, with $J = [0, T]$ some time interval, and $X_t = X(t)$ the value of a sample path (random function, or trajectory) at time t. The probability measure associated with the process is specified once $P(X(t_1) \leqslant x_1, \ldots, X(t_n) \leqslant x_n)$ is defined for all reals x_j, $j = 1, \ldots, n$ and any selection of $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in J$. There are two ways of looking at X_t: for a given trial, one may either refer to the selection of a sample path, or one refers to $X(t)$ as a random variable at time $t \in J$ (see e.g. Papoulis, 1965; Wong, 1971). It will be clear from the context which interpretation is referred to.

The trajectories of the process X_t will depend on the value of c, the contrast or amplitude of the stimulus, so one may write $X(t, c)$ for each trajectory of X_t. It will be assumed that the expected value $g(t, c) = E(X(t, c))$ of $X(t, c)$ exists for all $t \in J$. Let $\xi(t, c) = X(t, c) - g(t, c)$. For the purpose of this paper it is useful to write

$$X(t, c) = g(t, c) + \xi(t, c), \tag{1}$$

see, however, the remark below. It will further be assumed that for $c = 0$, i.e. when no stimulus is presented,

$$g(t, 0) = \mu \quad \text{a constant for all } t \in J. \tag{2}$$

As a special case one may have $\mu = 0$.

Remark. One may be tempted to interpret Eq. (1) as representing additive noise. However, this is not necessarily implied, since $\xi(t)$ is simply the difference between $X(t)$ and $E[X(t)] = g(t)$. As already mentioned in the Introduction, the noise is considered to be multiplicative by some authors if $\sigma(t) \propto g(t)$, where $\sigma^2(t)$ is the variance of X at time t.

The notions of TPS and TPD are formally characterised as follows:

1. *TPS*: Let the random variable X^+ be defined as

$$X^+ = \max_{t \in J} [X(t)] = \max_{t \in J} [g(t, c) + \xi(t)]. \tag{3}$$

   The stimulus is detected if $X^+ > S$, S being some internal threshold, and detection is said to be by *Temporal Probability Summation* (TPS). X^+ may also be called *decision variable*.

2. *TPD*: Let $g_{\max}(c) = \max_{t \in J} g(t, c)$. Detection is said to be by *Temporal Peak Detection* (TPD) if the probability of detection $\psi(c)$ is determined, apart from random effects, only by the value of $g_{\max}(c)$.

Comments:

1. *Activity*: The stochastic process X_t will be further specified when particular approximations of the psychometric function for detection by TPS will be considered.

2. *Single channel*: The activity is assumed to be represented by a single sample path X_t, so the considerations are restricted to detection by a single channel, or, equivalently, to a single population of neurons, where X_t represents the population activity. The results may, however, be generalised to detection by more than one channel.

3. *The definition of $J = [0, T]$*: The stimulus is presented at time $t = 0$, and has duration T_s; since the response to the stimulus may exceed the time T_s (e.g. the impulse response covers a larger interval of time than the duration of a brief pulse), the value of T has to be chosen larger than T_s, with the possible exception of a step stimulus, where one may choose $T = T_s$. The beginning and the end of the interval will be signalled to the subject. There are two possibilities: (1) J may be identified within an experimental trial, i.e. the subject has to respond before the trial has ended at time T, or (2) the subject has to respond after the time T, without further temporal restriction. This requires the instruction to the subject to respond only to the activity during the interval J. In case of (1) the need to respond before the trial has ended implies that the subject has to refer to a temporal threshold $T_0 < T$: if the activity $X(t)$ has exceeded the threshold S within $[0, T_0]$ the response will be "yes", otherwise the response will be "no". The subject has to determine T_0 itself and the problem is that T_0 will be a random variable. To cover this case, the distribution function of T_0 has to be taken into account and the interpretation of the data depends on the type of this distribution. Since this distribution is not known the interpretation of the data becomes exceedingly difficult, if not impossible. This is the reason why, in condition (1), there is no further temporal restriction for the response; if there were such a restriction, i.e. a further interval within which the subject has to respond, the subject would again have to refer to a temporal threshold preceding the end of the second interval. Consequently, case (2) will be assumed in the following.

4. *General form of the psychometric function for TPS*: Let $\psi : \mathbb{R} \to (0, 1)$ be the psychometric function. Since detection is postulated to occur if $\{X^+ > S\}$, ψ is assumed to be given by $\psi(c) = 1 - P(X^+ \leqslant S | c)$. In order to keep the expressions for ψ as simple as possible the probability of guessing that a stimulus was shown will be neglected in this paper.

Let τ be the time up to the first crossing of the level S by a sample path X_t; τ is known as waiting time. Thus $P(X^+ > S) = P(\tau \leqslant T) = F_\tau(T)$, where F_τ is the distribution function of τ. So detection may occur at any $\tau \in J$, which is the assumption of detection by TPS. A very general expression for ψ may be given making use of the notion of hazard-function. Since ψ is also a function of T, one may write $\psi(c, T)$. Let $\phi(t, c) \, dt = P(\tau \in (t, t + dt] | \tau > t, c)$; here, $\phi(t, c) \geqslant 0$ is the hazard-function associated with the process X_t,

and $\phi(t, c) = f_\tau(t)/(1 - F_\tau(t))$, so that

$$\psi(c, T) = 1 - \exp\left[-\int_0^T \phi(t, c) \, dt\right], \qquad (4)$$

(Papoulis, 1965, p. 109), where the probability of guessing that a stimulus was presented has been neglected. Expression (4) makes explicit that ψ depends on the value of T, the length of an experimental trial, when detection is by TPS. In fact, the case that ψ does not depend upon T will be encountered when white noise is assumed (cf. Eq. (14) in Section 2.2) and it is readily shown that this independency implies an inconsistency, as is easily seen: $\int_o^T \phi(t) \, dt = k$ independent of T implies $dk/dT = \phi(T) \equiv 0$, i.e. $\phi(t) = 0$ for all t, and $\phi(t) = f(t)/(1 - F(t)) \equiv 0$ implies $f(t) \equiv 0$, which does not make sense.

To keep the expressions simple we will write simply $\psi(c)$ instead of $\psi(c, T)$.

5. *Approximating TPD*: The question is whether detection by TPS on the one hand and detection by TPD on the other are completely different modes of detection, or whether TPS is actually the general mode of detection, with TPD representing an approximation that may be assumed under certain special conditions. This question is relevant for the discussion of the neural mechanisms that relate the neural activity generated by the stimulus and the process of actually perceiving the stimulus.

Let us note first that in order to define the probability of detection in case of TPD, a random variable η has to be introduced together with the postulate that detection occurs whenever $g_{max} + \eta > S$, S some internal threshold. The psychometric function is then given by

$$\psi(c) = 1 - P(\eta \leqslant S - g_{max}(c)). \qquad (5)$$

Alternatives to this definition and the discussion of the meaning of η will be deferred to a separate paper, since the following considerations apply to them equally well. Suppose detection is generally by TPS. The special conditions mentioned above are defined (i) by the temporal form of the stimulus, e.g. a brief pulse or a step function, (ii) the spatial form of the stimulus, which may determine the type of channel by which the stimulus is detected (e.g. transient or sustained), and (iii) the value of the internal threshold S relative to the values of parameters that characterise the noise.

Suppose the stimulus is a brief pulse and the corresponding mean response of the detecting channel is sufficiently peaked. It is intuitively plausible and will be demonstrated in the accompanying paper, that if detection is actually by TPS, the assumption of TPD in combination with the perturbation method suggested by Roufs and Blommaert (1981) still yields results that closely resemble those that were actually found by these authors. The situation is different, however, when the stimulus is a temporal step function and the tapped channel is of sustained type. If detection is by TPS, the probability of detection increases with T for $T > t_{max}$, in

contrast to the case of detection by TPD, where no such increase should be observed.

Let $F_\phi(T) = \exp(-\int_0^T \phi(t)\,dt)$ and $H(T) = P(\eta \leqslant S - g^+ | T)$, where $g^+ = \max_{t \in [0,T]} g(t)$; if $T > t_{\max}$, $g^+ = g_{\max}$. Let, T_s be the duration of the stimulus and let $T = T_s < t_{\max}$. Then $F_\phi(T)$ and $H(T)$ will decrease with T as T approaches $t_{\max}$. For $T > t_{\max}$ $H(T)$ remains constant. Suppose that there exist values of parameters defining ϕ such that:

$$F_\phi(T) \approx H(T) = h_0 \quad \text{for all } T > t_{\max}, \tag{6}$$

h_0 some constant. Then

$$\frac{dF_\phi(T)}{dT} \approx \frac{H(T)}{dT} = 0 \quad \text{for all } T > t_{\max}. \tag{7}$$

Now

$$\frac{dF_\phi(T)}{dT} = F_\phi(T)\phi(T) \approx 0 \tag{8}$$

only if $\phi(T) \approx 0$, which implies $f(t) \approx 0$ for all $t > t_{\max}$; this is a similar argument as in (4). Here, (8) implies that in case of detection by TPS the hazard-function ϕ is such that (almost) no detection can occur within the interval $(t_{\max}, T]$. However, if the stimulus is a temporal step $g(t)$ will, in general, not be equal to zero for $t \in (t_{\max}, T]$, i.e. the postulate $f(t) \approx 0$ for $t \in (t_{\max}, T]$ does not make any sense, in particular since one has to postulate $f(t) > 0$ for $t \leqslant t_{\max}$ although $g(t) \leqslant g_{\max}$ on this interval. In other words, (8) is scarcely compatible with the notion of TPS. The argument does not refer to any specific parameter values and is thus independent of such values. It follows that if detection is by TPS, then there is no approximation in terms of TPD, and if detection is by TPD, then there is no representation of the detection process in terms of a TPS-model.

6. *False alarms*: Suppose ψ is defined as in (4). For $c = 0$ the probability of a false alarm is given by $\psi(0)$. If $\psi(0) > 0$, it follows that $\phi(t, 0)$ is not equal to zero for all $t \in [0, T]$, and the probability of a false alarm will depend on the duration T of a trial. If $\phi(t, 0) = 0$ for all $t \in J$, then $\psi(0) = 0$ and detection is said to be "high-threshold". Conversely, if detection is high-threshold in a rigorous, not approximative sense, then $\int_o^T \phi(t, 0)\,dt = 0$ and because generally $\phi(t, 0) \geqslant 0$ for all t it follows in this special case that $\phi(t, 0) = 0$ for all t.

7. *Guessing*: The assumption that the probability of guessing may be neglected was only introduced to keep the expression for the psychometric function (4) as simple as possible. In this paper, the focus is on properties of the hazard-function ϕ.

2.2. The case of stationary and δ-correlated noise

In this section, psychometric functions will be derived by assuming δ-correlated (white) noise. The main characteristic of this type of noise is that its autocorrelation $R(\tau)$ is proportional to the Dirac function $\delta(t)$, with $\delta(t) = 0$ for

$t \neq 0$, $\delta(0) = \infty$ and $\int_{-\infty}^\infty \delta(t)\,dt = 1$. One may write $R(\tau) = Q(\tau)\delta(\tau)$, with Q some function of τ; in particular, one may choose $Q(0) = \sigma^2 < \infty$, see Bucy–Joseph (1968), Jazwinkski (1970), Dayan and Abbott (2001), Honerkamp (1994), Gerstner and Kistler (2002, p. 174). The power spectrum of δ-correlated noise is known to be constant (for a derivation see Dayan & Abbott, 2001).

This concept of white noise will be adopted since noise defined in terms of the Weibull distribution with finite variance will be considered. The expression δ-correlated noise will be employed because the proportionality of $R(\tau)$ to $\delta(\tau)$ matters in the following.

In order to arrive at an explicit expression for $P(X^+ \leqslant S)$, an assumption concerning the probability measure associated with $\{\xi_t, t \in J\}$ has to be made. Two cases will be considered: for any $t \in J$, $\xi(t)$ is (i) Weibull distributed or (ii) Gaussian distributed. The first case was (implicitly) adopted by Watson (1979). However, it will be shown that some of the implications of the assumption of δ-correlated noise do not depend on the type of distribution. One possible approach to the problem of finding expressions for detection by TPS is to formulate a Fokker–Planck equation (see e.g. Lindenberg & Seshadri, 1979). Although this approach is very interesting with respect to the detailed structure of the stochastic aspects of the neuronal activity, its intricacies seem to outweigh the gain of insight with respect to detection data, unless considerable simplifications are introduced. An alternative approach to the derivation of expressions for $P(X^+ \leqslant S)$ is to make use of results of the theory of extreme values. This approach will be followed in this paper.

Let the interval $J = [0, T]$ be divided into n subintervals $J_i = (t_{i-1}, t_i]$ of equal length $\Delta t = T/n$, see Fig. 1. Let the random event $\bar{D}_i$ be defined by $\bar{D}_i = \{$no detection $within J_i\}$, $i = 1, \ldots, n$. Let

$$X_i^* = \max_{t \in J_i} X(t) \tag{9}$$

be the random variable representing the maximum of the activity $X(t)$ within J_i. There exists a $t_i^* \in J_i$ such that $X_i^* = g(t_i^*) + \xi(t_i^*)$. Then

$$P(\bar{D}_i) = P(X_i^* \leqslant S) = P(\xi(t_i^*) \leqslant S - g(t_i^*)) = P(\xi(t_i^*) \leqslant x_i^*),$$
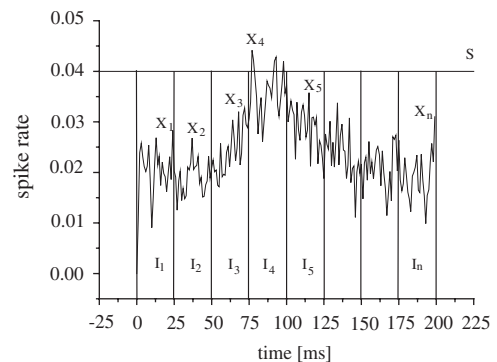

Fig. 1. Maxima in subintervals.

with $x_i^* = S - g(t_i^*)$, and for $n \to \infty$ it follows that $X_i^* \to X(t_i) = g(t_i) + \xi(t_i)$. Let us consider the following assumptions:

1. for all $n \in \mathbb{N}$,

$$P(\bar{D}_i \cap \bar{D}_j) = P(\bar{D}_i)P(\bar{D}_j), \quad i \neq j, \tag{10}$$

2. and

$$P(\bar{D}_i) = \begin{cases} \exp[-(S - x_i^*)^\beta], & S - x_i^* \geqslant 0, \quad \text{(a)}, \\ \Phi(S), & \text{(b)}. \end{cases} \tag{11}$$

In case of (11a), $P(\bar{D}_i)$ is assumed to be defined by the Weibull distribution, in which case one may speak of Weibull δ-correlated noise. Eq. (10) is meant to hold for all values of n, i.e. also for $n \to \infty$ and consequently $\Delta t \to 0$. The postulated independence of the $\bar{D}_i$ for all n is equivalent to postulating that the random variables $X(t)$, i.e. the values of X at any time t, are all independent. This means that the autocorrelation function of the stochastic process that represents the noise satisfies $R(\tau) = 0$ for $\tau \neq 0$. Consequently, the noise is δ-correlated, and if $R(0) = \sigma^2 < \infty$ the noise is white in the sense of Bucy–Joseph (1968) and Jazwinski (1970). In case of (11b), $\Phi(S) = P(X_i^* \leqslant S)$, X_i^* is Gaussian distributed and one may speak of Gaussian δ-correlated noise. A similar argument holds for the Gaussian distribution with finite variance. Psychometric functions for δ-correlated noise are characterised in the following.

Theorem 2.1. *Suppose assumptions* (10) *and* (11) *hold. If* $\{\xi_t, t \in J\}$ *is defined as Weibull δ-correlated noise, the psychometric function is given by*

$$\psi_{\mathrm{W}}(c) = 1 - \exp\left[-\frac{1}{T}\int_0^T (g(t) - S; c))^\beta \, dt\right], \quad g(t) - S \geqslant 0. \tag{12}$$

If $\{\xi_t, t \in J\}$ *is defined as Gaussian δ-correlated noise, then*

$$\psi_{\mathrm{G}}(c) = 1 - \exp\left[-\frac{1}{T}\int_0^T e^{-(S - g(t,c))} \, dt\right]. \tag{13}$$

Proof. See Appendix, Section A.1.

Implications:

1. *Probability of false alarms*: Suppose no stimulus is presented in a trial, so $g(t) = 0$ for all $t \in [0, T]$. Eqs. (12) and (13) then immediately imply that

$$\psi(0) = 1 - \begin{cases} \exp(-(-S)^\beta), & S \leqslant 0, \text{ Weibull}, \\ \exp(-e^{-S}) & \text{Gauss}. \end{cases} \tag{14}$$

So the probability of a false alarm does not depend on the duration T of the trial. However, the observation in Comment 4 in Section 2.1 implies $\phi(t, 0) = 0$ and therefore $f(t, 0) = 0$ for all t in both cases, independent of the value of S. So (14) implies an inconsistency,

resulting from the unjustified assumption of δ-correlated noise.

2. *Approximation of TPD*: Since neither $\phi_{\mathrm{W}}(t)$ nor $\phi_{\mathrm{G}}(t)$ are proper hazard-functions, it does not make sense to discuss the possibility of any approximation of TPD in case of δ-correlated noise.

The model of Watson (1979): Since in visual psycho-physics this model appears to have acquired the status of a canonical model, a few remarks may be in order. Watson assumed g to be linear in c so that $g(t, c) = ch(t)$, with $h(t)$ the unit response of the responding channel at time t, and derived the psychometric function

$$\psi_{\mathrm{W}'}(c) = 1 - (1 - \gamma)\exp\left(-\int_{-\infty}^{\infty} |ch(t)|^\beta \, dt\right), \quad c \geqslant 0, \ \beta > 0. \tag{15}$$

Watson based his derivation on the assumptions (i) that the probability of no detection in the ith subinterval is given[3] by $P(\bar{D}_i) = \exp[-|g_i|^\beta]$, so he assumed the existence of some rectifying process which is not necessarily required in case of the Weibull distribution, and (ii) that the detection events in subintervals are independent for all n, so he postulated δ-correlated noise. The probability of not detecting the stimulus equals $P_n(\bar{D}) = \exp(-\sum_i |g_i|^\beta)$, with $g_i = ch(t_i)$. His claim that for $n \to \infty$ $P_n(\bar{D})$ converges towards $\exp[-c^\beta \int_{-\infty}^{\infty} |h(t)|^\beta \, dt]$ may find a certain justification insofar as the limiting distribution of Weibull-distributed random variables is again of Weibull-type. However, omitting the required re-scaling of the Weibull variables (see Appendix, Section A.1.1) implies, for instance, that the factor $1/T$ does not appear as in (12), and the need to introduce the rectification of g. Since $h(t)$ is usually defined for $t \geqslant 0$ the lower limit $-\infty$ in (15) may be replaced by 0. The upper limit ∞ in (15) may be replaced by the maximal value of t for which $|g(t)| > 0$, so (15) complies with (4) when $\phi(t) = |ch(t)|^\beta$ with $h(t) = 0$ if t is larger than a certain value T.

The fact that the derivation of Eq. (15) may be criticised does not yet mean that no other, correct derivation of this expression for a psychometric function exists. If such a derivation exists, the hazard-function would be $\phi(t) = |g(t)|^\beta$, allowing for the approximation of the TPD-case when β is large. Watson (1982) argued that large values of β, say $\beta > 7.5$, are usually not observed and that therefore the assumption of TPD is not meaningful. This conclusion is clearly not justified, since there is no reason to consider (15) as the canonical psychometric function.

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[3]This corresponds to (11a): $g_i = g(t_i^*)$ is a value of the deterministic response g for some value of $t_i^* \in J_i$. With $X_i^* = |g_i| + \xi(t_i^*)$ one has $P(\bar{D}_i) = P(X_i^* \leqslant S) = P(\xi(t_i^*) \leqslant x_i^*)$ with $x_i^* = S - |g_i|$. It follows that $P(\bar{D}_i) = \exp[-|g_i|^\beta] = \exp[-(S - S + |g_i|)^\beta] = \exp[-(S - x_i^*)^\beta]$ for $S - x_i^* \geqslant 0$.

2.3. The case of stationary and coloured Gaussian noise

The following theorem contains the main result of this paper:

Theorem 2.2. *Suppose the following conditions hold*:

1. *The sample paths of the stochastic process representing the sensory activity are assumed to be given by $x(t) = g(t, c) + \xi(t)$, where $\xi(t) = \xi_t$ is a differentiable sample path of a stationary, Gaussian process $\mathbf{N} = \{\xi_t, t \in J\}$, with $E(\xi(t)) = 0$, for all t, so that $g(t, c) = E(X(t, c))$ is the mean activity at time t. Furthermore, $Var(\xi(t)) = \sigma^2$ is independent of t.*
2. *The autocorrelation of ξ_t may be expanded into a Taylor series*

$$R(\tau) = 1 + \tfrac{1}{2} R''(0)\tau^2 + o(\tau) = 1 - \tfrac{1}{2}\lambda_2\tau^2 + o(\tau^2), \qquad (16)$$

where $\lambda_2 = -R''(0) = -d^2 R(\tau)/dt^2|_{\tau=0}$ is the second spectral moment of ξ_t, and $o(\tau)$ represents the sum of all higher terms which approaches 0 faster than τ.

Then, for $S \to \infty$, the psychometric function is given by

$$\psi(c) = 1 - \exp\left[-\frac{\sqrt{\lambda_2}}{2\pi} \int_0^T \exp\left(-\frac{1}{2}(S - g(\tau, c))^2\right) d\tau\right], \qquad (17)$$

where the values of S and g are again expressed in units of σ.

Proof. See the Appendix, Section A.2.

Comments:

1. *Type of noise process*: It had to be postulated that the noise has differentiable sample paths. This assumption implies that $\mathbf{N} = \{\xi_t, t \in J\}$ is not a Markov process, i.e. $\mathbf{N}$ is not the Ornstein–Uhlenbeck process often assumed in models formulated in terms of stochastic differential equations. This need not be seen as a restriction of generality; on the contrary, Markov processes generally represent approximations of actual, physical processes (Arnold, 1973, Chapter 10).
   The postulated stationarity of the noise process may be viewed as being more critical than the question whether the sample paths are differentiable or not. As the review of neurophysiological results shows, there is substantial evidence that the noise, as measured in single neurons, is multiplicative. It should, however, be recalled that at the site where the decision is made whether a stimulus was presented or not the stimulus is processed most likely by cell assemblies (Silberberg et al., 2004; Harris, 2005); detection is a response to a—except for false alarms—stimulus generated modulation of activity of the assembly. Many nonlinear systems can be approximated by linear ones for small perturbations, so one may here assume that the postulated stationarity represents a reasonable approximation. Gorea and Sagi (2001) found that in contrast discrimination tasks—the sort of task

considered in this paper—the performance of their subjects was limited (i) by a transducer nonlinearity, and (ii) by an almost constant noise. Since linearity of the detecting channel is not postulated in Theorem 2.2, the results of Gorea and Sagi (2001) may be taken as experimental support for the simplifying assumption of stationary noise.

2. *The density function of the decision variable*: Recall that the psychometric function was defined in Comment 4 as $\psi(c) = 1 - P(X^+ \leqslant S|c)$, X^+ being the decision variable, X^+ as defined in Eq. (3). Let $F_{X^+}^{\mathbf{sn}}(x) = P(X^+ \leqslant x|c \neq 0)$ and $F_{X_+}^{\mathbf{n}}(x) = P(X^+ \leqslant x|c = 0)$. For the corresponding density functions $f_{\mathbf{sn}}(x)$ and $f_{\mathbf{n}}(x)$ one finds

$$f_{\mathbf{sn}}(x) = F_{X^+}^{\mathbf{sn}}(x) \frac{\sqrt{\lambda_2}}{2\pi} \int_0^T (x - g(t, c))$$

$$\times \exp\left[-\frac{1}{2}(x - g(t, c))^2\right] dt, \qquad (18)$$

$$f_{\mathbf{n}}(x) = F_{X^+}^{\mathbf{n}}(x) \frac{xT\sqrt{\lambda_2}}{2\pi} \exp\left(-\frac{1}{2}x^2\right). \qquad (19)$$

Explicit, analytic expressions for the expected value $\mu = E(X^+)$ and the variance $\sigma^2 = Var(X^+)$ of X^+ are not known; the values of these statistics have to be found numerically. To calculate them for the case that a stimulus was presented the mean value function g must be known. Obviously, μ and σ^2 depend on the complete course of g during the trial. The "noise distribution", i.e. the distribution of X^+ when no stimulus was shown, and the corresponding "signal + noise distribution" will have different variances, see Fig. 2 for an illustration. There, the signal s is a rectangular pulse of duration 50 ms, the duration of a trial is 2000 ms, and the response g is the convolution $g = s * h$ with the impulse response
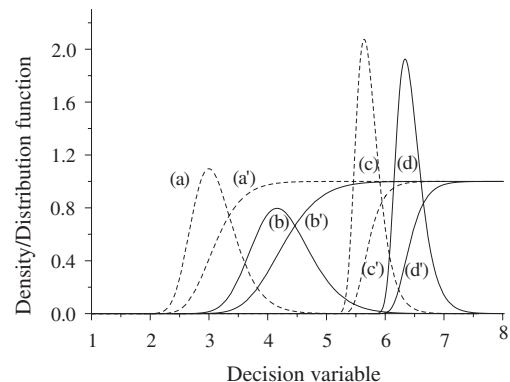

Fig. 2. Density and distribution functions for noise alone and signal-plus-noise. Signal: a rectangular pulse of 50 ms duration. $\lambda_2 = 1$: (a) and (a′) density and distribution function, noise alone, (b), (b′) density and distribution function, signal + noise. $\lambda_2 = 10^{10}$: (c), (c′) density and distribution function, noise alone, and (d), (d′) density and distribution function, signal + noise. A larger λ_2-value implies larger expected values, but the variances of the decision variable will be reduced so that corresponding d'-values will be larger than those for smaller λ_2-values (cf. Section 2.5).

$h(t) = \alpha(at)^p \exp(-at)$, the impulse response of a sustained channel as determined by Roufs et al.

3. *The meaning of λ_2*: The spectral density of the noise process is given by the Fourier transform of $R(\tau)$, i.e. $S(\omega) = \int_{-\infty}^{\infty} R(\tau) \exp(-i\omega\tau) \, d\tau$, so $R(\tau) = \int_{-\infty}^{\infty} S(\omega) \exp(i\omega\tau) \, d\omega/2\pi$, $i = \sqrt{-1}$. The first two derivatives of $R(\tau)$ are then given by

$$R'(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega) S(\omega) e^{i\omega\tau} \, d\omega,$$

$$R''(\tau) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 S(\omega) e^{i\omega\tau} \, d\omega,$$

and $R'(0) = 0$ since $S(\omega)$ is an even and ω is an odd function of ω. Then

$$\lambda_2 = -R''(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 S(\omega) \, d\tau,$$

which explains the term "second spectral moment". If the noise is white one has $R(\tau) \propto \delta(\tau)$ and the spectral density $S(\omega) = S_0$ is constant, implying $\lambda_2 = \infty$. On the other hand, if $R(\tau) = R_0$, $S(\omega) \propto \delta(\omega)$ and $\lambda_2 = 0$. For any realistic noise process neither $R(\tau) \propto \delta(\tau)$ nor $R(\tau) = R_0$, a constant, will hold, so $0 < \lambda_2 < \infty$. A large value of λ_2 reflects a noise with fast fluctuations, whereas a small value implies a slowly fluctuating process. The larger the value of λ_2, the more the distribution of X^+ and the corresponding density function is shifted to the right, and the smaller the value of $\sigma_x^2 = Var(X^+)$, i.e. the variance of the decision variable X^+, cf. Fig. 2. This is a consequence of the fact that the decision variable is the maximum of the random function X_t within the interval $[0, T]$: the larger the value of λ_2, the faster the trajectories ξ_t of the noise fluctuate and the larger the chance that the detection level S is reached.

While large values of λ_2 mean that the noise approximates white noise, no value of λ_2 can be found for which expression (17) for ψ can assume the form of ψ_G in (13). So for any finite value of λ_2 characteristic properties of ψ as defined by (17) remain invariant. For $\lambda_2 = \infty$ the expression becomes meaningless, which reflects the fact that white noise in the strict sense does not exist. In other words, starting with the assumption of white noise leads automatically to inadequate expressions for the psychometric function.

4. *False alarms and the corresponding first-passage time distribution*: The hazard-function corresponding to (17) is given by

$$\phi(t) = \frac{\sqrt{\lambda_2}}{2\pi} \exp\left(-\frac{1}{s}(S - g(t, c))^2\right). \tag{20}$$

The probability of a false alarm is given by

$$\psi(0) = 1 - \exp\left(-\frac{\sqrt{\lambda_2}}{2\pi} T e^{-S^2/2}\right), \tag{21}$$

in this case, the hazard-function is defined by

$$\phi(t, 0) = \phi_0 = \frac{\sqrt{\lambda_2}}{2\pi} e^{-S^2/2}. \tag{22}$$

Obviously, ϕ_0 is a constant and it follows that the corresponding first-passage time distribution is exponential:

$$P(\tau \leqslant t | S, c = 0) = 1 - \exp(-\phi_0 t). \tag{23}$$

The expectation and variance of an exponential distribution are given by $E(\tau) = 1/\phi_0$ and $\sigma^2 = 1/\phi_0^2$, respectively, so a large value of the second spectral moment λ_2, relative to that of S, will imply a correspondingly small value of the mean waiting time and the variance, while a large value of S, relative to that of λ_2, will imply a large value of $E(\tau)$ and σ^2.

For a given value of t, in particular for $t = T$, and as a function of the parameter S, however, $P(\tau \leqslant t | S, c = 0)$ is given by the extreme value or double exponential distribution, which is to be expected for the extremes of Gaussian variables; note that this limiting distribution is of the same type as that for independent variables.

5. *The effect of the noise*: The notion of the effect of the noise is often associated with the steepness of the psychometric function, and if the psychometric function is defined by a Weibull-function, its steepness is controlled by the parameter β. A large value of β implies a small variance of the decision variable relative to the amplitude of the deterministic response g. If the psychometric function is given by (17), the characterisation of the effect of noise is more complex. The probability of a false alarm is given by (21) meaning that $\psi(0)$ is determined by the values of S and λ_2. To the effect that λ_2 reflects a physiological constant the value of $\psi(0)$ depends on that of S, and the value of S is possibly determined by the experimental conditions (instructions concerning the validity of detection judgments, pay-off conditions, etc.). In order to arrive at a specific value of $\psi(0)$ the subject has to "choose" that value of S that implies a certain value of $\psi(0)$, and the value of S has to be chosen relative to that of λ_2,

$$S = \sqrt{\log \lambda_2 + k_0}, \tag{24}$$

with $k_0 = 2[\log T - \log(-2\pi \log(1 - \psi(0)))]$. The steepness is not defined by a single parameter, but by the combination of S and λ_2, which again specifies the value of $\psi(0)$. For a fixed value of λ_2 the psychometric function may be more or less steep, depending on the value of $\psi(0)$ induced by a choice of S. Further,

$$\frac{dS}{d\lambda_2} = \frac{1}{\lambda_2 \sqrt{\log \lambda_2 + k_0}} > 0, \tag{25}$$

i.e. S will increase with λ_2, but at a decreasing rate. Fig. 3 illustrates the effect of the noise. Panel (a) shows psychometric functions as a 3D-plot for a false alarm rate $\psi(0) = .01$. For small values of $\lambda_2 = 10^n$, $n = 0$,
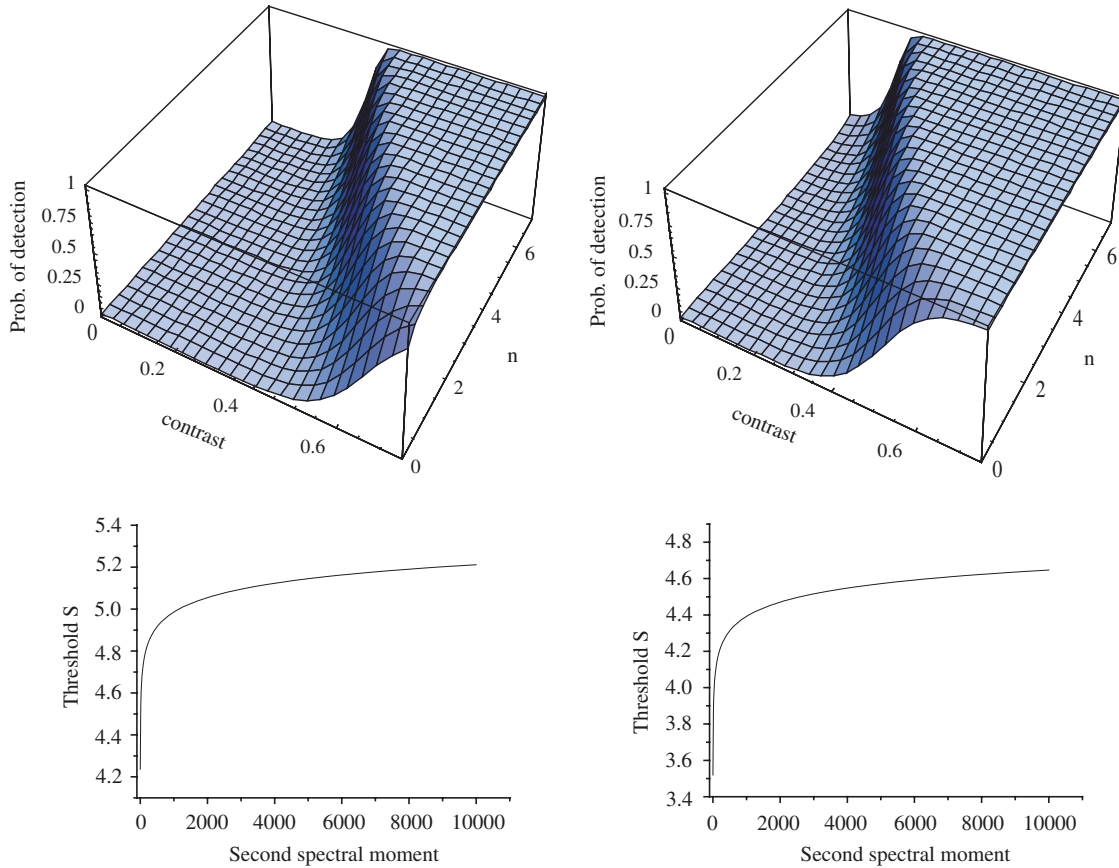
Fig. 3. The effect of the noise: psychometric functions for different values of $\psi(0)$, S and λ_2, g is given by the impulse response as determined by Roufs and Blommaert (1981), i.e. $g(t) = c\alpha(at)^p \exp(-at)$, with $a = 1/12.66$, $p = 3$, $\alpha = 3.7$, trial duration $T = 500$ ms: (a) $\psi(0) = .01$; (b) $\psi(0) = .15$. The values of λ_2 are given by $\lambda_2 = 10^n$. (c) and (d) show the plots (24) of S versus λ_2 for $\psi(0) = .01$ and $\psi(0) = .15$, respectively. See text for further explanation.

$n = 1$, etc. The psychometric functions have a smaller slope and are shifted to the right on the contrast axis, compared to the psychometric functions for larger values of λ_2. In (b), the false alarm rate equals $\psi(0) = .15$. The picture is qualitatively similar, although now the psychometric functions are shifted to the left compared to those in (a), and their slope is altogether greater. For $\lambda_2 > 10^3$ the threshold parameter S increases only very slowly with λ_2.

Note that $\psi(0)$, as defined in Eq. (21), depends on the duration T of an experimental trial, whereas the expressions (14) for $\psi(0)$, based on the assumption of δ-correlated noise, do not (see also Comment 4 with respect to (4)); in (15) $\psi(0) = \gamma$ since Watson's model is a high-threshold model. It remains an empirical question as to which extent the probability of a false alarm does indeed depend on the duration T of a trial. A subject may refer to something like a subjectively defined standard duration (for example, an "attentional" window) of a trial with respect to which the possible occurrence of a stimulus is evaluated. Such aspects of the decision behaviour are not part of the definition of psychometric function as discussed in this paper.

6. *The possible range of values for S*: The approximation (17) is valid for $S \to \infty$. This means that the value of S

has to be "large",[4] e.g. $S > 3$. As said in the preceding comment, a value of $\psi(0)$ defines a set of pairs (λ_2, S). Although one finds that smaller values of λ_2 require smaller values for S, even for $\lambda_2 < 1$, values of $S > 3\sigma$ are found (here: $\sigma = 1$), so one is always on the safe side for the approximation to hold. One arrives at the same conclusion when (17) is compared with Ditlivsen's approximation (26), presented in the following section, which does not depend on "large" values of S.

2.4. Ditlivsen's approximation

In this section, a result of Ditlivsen (1971) concerning level-crossing processes will be presented; the approximation (26) will be referred to as Ditlivsen's approximation. This approximation is not based on extreme-value statistics and neither requires the side condition $S \to \infty$, i.e. "large" values of S, nor the assumption of stationary noise. Ditlivsen's approximation will be employed as a test of the goodness of the mathematically more straightforward extreme-value approximation (17).

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[4]Actually, $S > 3\sigma$. However, $\sigma = 1$ in the derivation of the limiting distribution.

Theorem 2.3. *Let ξ_t be a Gaussian process with $E(\xi(t)) = 0$ and $E(\xi^2(t)) = 1$ for all $t \in [0, T]$. Suppose that the autocorrelation function of ξ_t satisfies $R(\tau) = 1 - \lambda_2\tau^2/2 + o(\tau^2)$ as $\tau \to 0$. Let $g'(t) = dg(t)/dt$, $f(u) = \exp(-u^2/2)/\sqrt{2\pi}$, $\Phi(u) = \int_{-\infty}^{u} f(t)\,dt$, $\gamma = \sqrt{\lambda_2}$, $b(t) = g'(t)/\sqrt{\lambda_2}$, and $a(t) = S - g(t)$. Then*

$$P(X^+ \leqslant S)$$
$$\approx \Phi(S)\exp\left(-\int_0^T [\gamma f(b(t)) + \Phi(b(t))g'(t)]\frac{f(a(t))}{\Phi(a(t))}\,dt\right)$$
(26)

and $\psi(c) = 1 - P(X^+ \leqslant S)$.

Proof. The proof can be found in Ditlivsen (1971) and is beyond the scope of this paper. It rests on a number of results on Gaussian processes with differentiable sample paths (cf. Cramér & Leadbetter, 1967), and its details are not required in the following. □

Like the approximation (17), Ditlivsen's expression (26) for the probability of a level crossing depends on the value of the second spectral moment λ_2. Compared to (17), the expression is mathematically more complicated. For $g(t) \equiv 0$ the waiting time (for a false alarm) is again exponentially distributed. Numerical comparisons show, however, that except for very small values of S, the approximation (17) on the one hand and Ditlivsen's on the other lead to practically identical results, which means that the extreme-value approximation is very robust.

2.5. The relation between values of d' and λ_2

In order to get a measure of sensitivity one usually computes the value of d' from estimates of the probability of detection; according to the standard assumptions of signal-detection theory, the noise- and the signal-plus-noise distributions are Gaussian with equal variance. These assumptions may well be violated when detection is by TPS, since then the distributions of the decision variable X^+ are not Gaussian, and the variances of X^+ differ under either condition. It may therefore be of interest to see in which way d', under TPS conditions, depends on the value of λ_2, and of course on the probability of a false alarm.

Let $X^+ = X^{\mathbf{n}}$ when no stimulus is presented, i.e. if $c = 0$, and $X^+ = X^{\mathbf{sn}}$ when a stimulus is presented, $c \neq 0$. Let $\bar{x}^{\mathbf{n}} = E(X^{\mathbf{n}})$ and $\bar{x}^{\mathbf{sn}} = E(X^{\mathbf{sn}})$, and let $\sigma_{\mathbf{n}}^2 = Var(X^{\mathbf{n}})$ and $\sigma_{\mathbf{sn}}^2 = Var(X^{\mathbf{sn}})$. The sensitivity for a stimulus, d', is generally defined as

$$d' = \frac{\bar{x}^{\mathbf{sn}} - \bar{x}^{\mathbf{n}}}{\sqrt{\frac{1}{2}(\sigma_{\mathbf{sn}}^2 + \sigma_{\mathbf{n}}^2)}}.$$
(27)

The yes–no-experiment: For a given value of λ_2, the values for $\bar{x}_{\mathbf{n}}$ and $\sigma_{\mathbf{n}}$ can be computed. The value of d' depends upon the stimulus amplitude c. This value was chosen such that $\psi(c) = .5$. The computation of $\psi(c)$ requires a value for the internal threshold S, which was chosen such that the probability of a false alarm equals .01. Values other than this, e.g. $\psi(0) = .1$, lead to qualitatively similar results. Fig. 4 shows the corresponding relationships between λ_2, d' and c.

According to plot (a), d' is a negatively accelerated function of λ_2; note that for each value of λ_2, the corresponding value of S was determined such that the probability of a false alarm equalled $p(fa) = .01$. For a given probability of a false alarm the value of d' increases with that of λ_2, meaning that the difference $\bar{x}^{\mathbf{sn}} - \bar{x}^{\mathbf{n}}$ of the means grows faster than the SD $\sqrt{(\sigma_{\mathbf{sn}}^2 + \sigma_{\mathbf{n}}^2)/2}$, but less so the larger the value of λ_2. The increase of d' with λ_2 implies a corresponding decrease of c; plot (b) shows the way c decreases with increasing value of λ_2. In Fig. 5, plots of the probability ψ of detection versus d' are shown for different values of λ_2 and different probabilities of a false alarm. The functions relating ψ and d' are the same for equal probabilities of a false alarm and different values of λ_2, although plots of ψ versus c, the amplitude of the stimulus, do depend on the value of λ_2. So, given a certain value of the probability of a false alarm, a certain value of ψ implies a certain value of d', regardless of the value of λ_2, i.e. although the values of $\bar{x}^{\mathbf{sn}} - \bar{x}^{\mathbf{n}}$ and $\sqrt{(\sigma_{\mathbf{sn}}^2 + \sigma_{\mathbf{n}}^2)/2}$ depend on λ_2, the quotient of the difference of the mean values and the SD is independent of λ_2. Additionally, the Gaussian curve, corresponding to the classical SDT-assumptions, has been added. Obviously, if detection is by TPS, the Gaussian assumption implies an underestimation of d', if $p(fa) = .05$, and an overestimation, if $p(fa) = .1$. It seems,
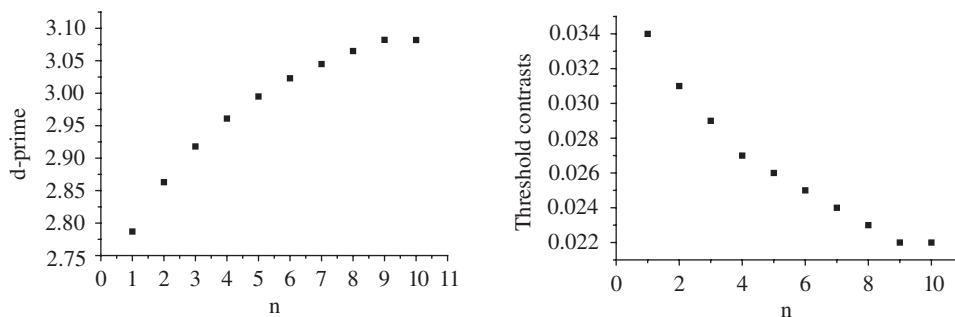

Fig. 4. Relationships (a) between $\lambda_2 = 10^n$ and d'; (b) λ_2 and c_0, for the probability of a false alarm equal to .01 and probability of detection $\psi(c_0) = .5$; abscissa represents values of λ_2, indicated by the corresponding values of $n = \log_{10} \lambda_2$. As in Fig. 2, the stimulus is a rectangular pulse of duration 50 ms, the duration of a trial is 2000 ms, and the response g is the convolution $g = s * h$ with the impulse response $h(t) = \alpha(at)^p \exp(-at)$.
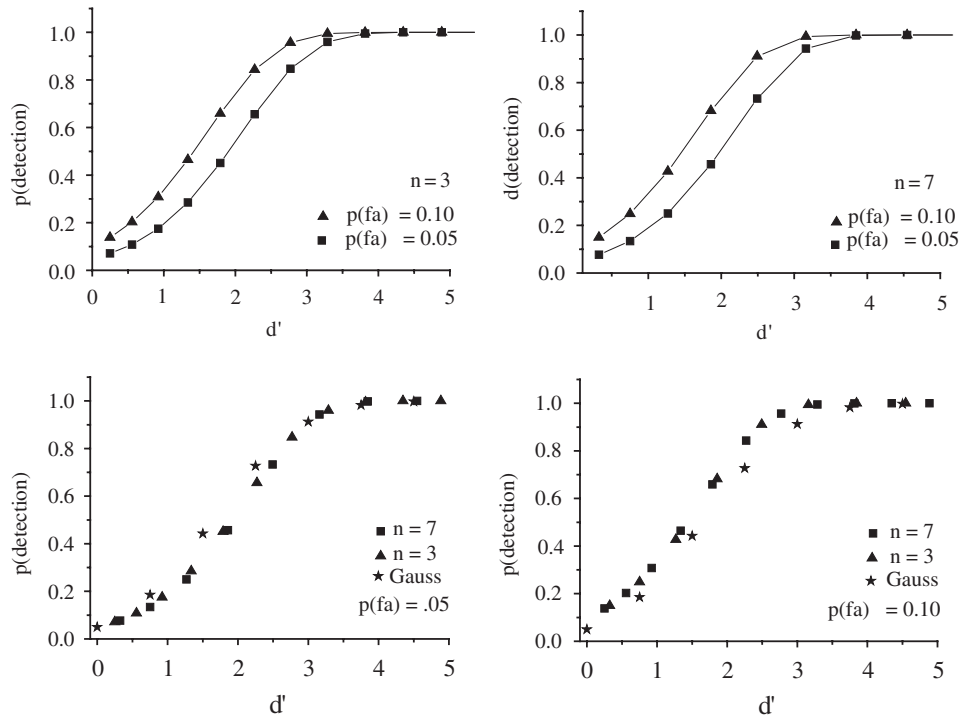
Fig. 5. Probability of detection versus d'. In (a), $\lambda_2 = 10^3$, in (b), $\lambda_2 = 10^7$, for probability of false alarms $p(fa) = .05$ and $p(fa) = .1$. In (c) and (d), plots for $\lambda_2 = 10^3$ or $\lambda_2 = 10^7$ but equal probabilities of false alarms were superimposed. Additionally, the corresponding points for the standard SDT-case (Gaussians with equal variance) were added.
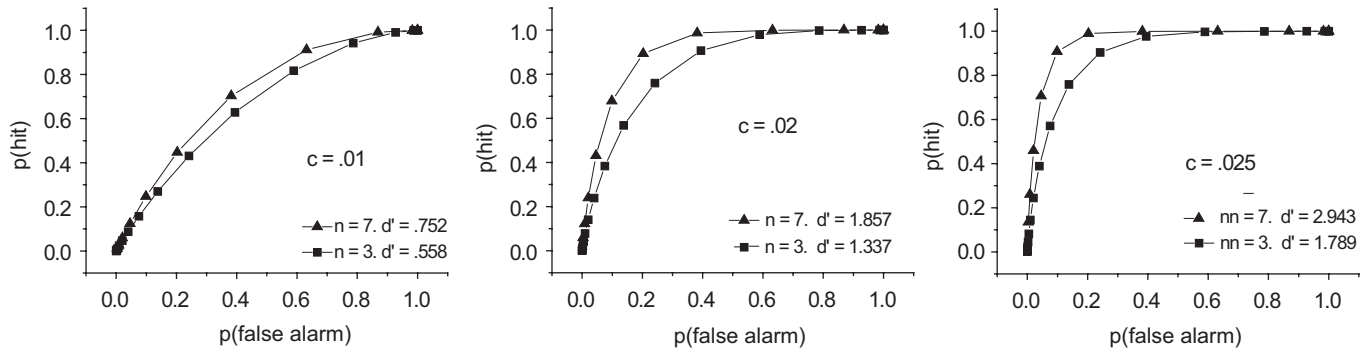

Fig. 6. ROC-curves for different values of $\lambda_2 = 10^n$ and d'; stimulus: a brief pulse.

however, that employing the standard SDT-model implies a relatively small error when detection is indeed by TPS, in particular for smaller values of the probability of a false alarm.

In Fig. 6, ROC-curves are shown for different values of the stimulus amplitude and different values of λ_2. Clearly, the greater the value of λ_2, the greater the sensitivity, as expressed in the corresponding d'-values.

The 2AFC experiment: The probability of a correct response is the probability of the event $\{X^{\mathbf{n}} < X^{\mathbf{sn}}\}$, which is given by

$$P(\text{correct}) = \int_{-\infty}^{\infty} f_{\mathbf{n}}(x) \int_{x}^{\infty} f_{\mathbf{sn}}(y) \, dy \, dx, \tag{28}$$

where $f_{\mathbf{n}}$ is the density of $X^{\mathbf{n}}$ and $f_{\mathbf{sn}}$ is that of $X^{\mathbf{sn}}$. Since $\int_{x}^{\infty} f_{\mathbf{sn}}(y) \, dy = 1 - F_{+}^{\mathbf{sn}}(x)$ it follows that $P(\text{correct}) =$

$\int_{-\infty}^{\infty} f_{\mathbf{n}}(x)(1 - F_{+}^{\mathbf{sn}}(x)) \, dx$, so that

$$P(\text{correct}) = 1 - \int_{-\infty}^{\infty} f_{\mathbf{n}}(x) F_{+}^{\mathbf{sn}}(x) \, dx. \tag{29}$$

Fig. 7(a) and (b) shows the sensitivity d' as a function of the contrast or amplitude c of the stimulus, for different values of λ_2; obviously, d' increases faster with c for larger values of λ_2. Fig. 7(b) shows the corresponding psychometric functions. The one defined by the larger λ_2-value is steeper. Note that the value of λ_2 is no longer coupled to a particular value of S, because of the integrals in (28) and (29). The higher the rate of the noise fluctuations, the higher the probability that the stimulus is detected. (c) and (d) show plots of ψ versus d' for two different stimuli, namely a brief pulse and a step. Obviously, the plots are invariant with respect to different values of λ_2, the stimulus
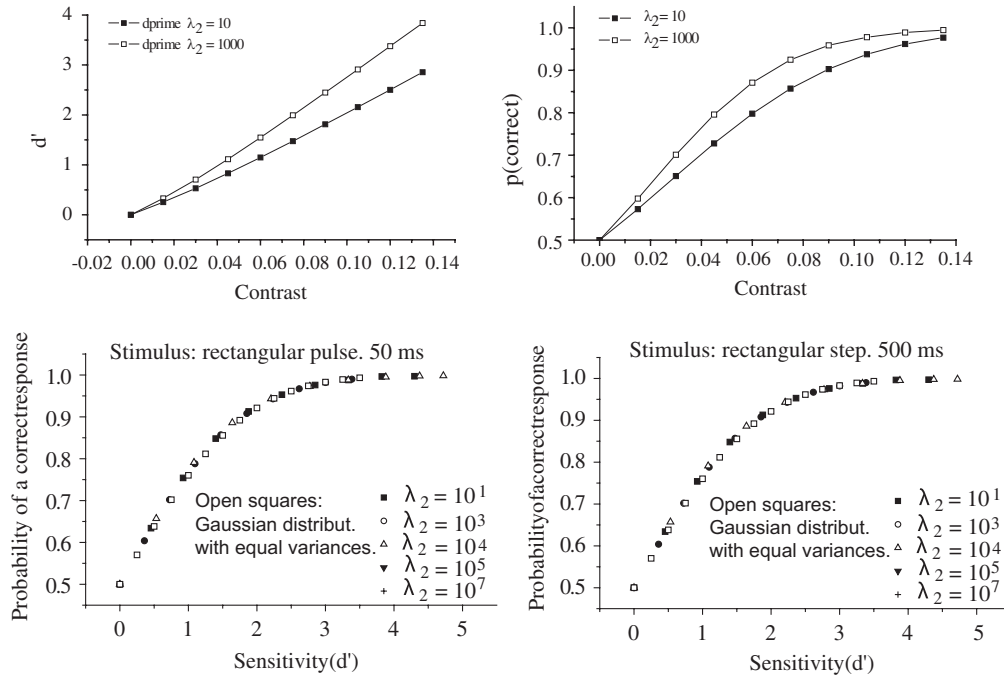
Fig. 7. The 2AFC-experiment: (a) d' as a function of contrast; (b) psychometric functions for $\lambda_2 = 10$ and 1000. The probability of a correct response as a function of d'; (c) for a brief pulse; (c') for a step input.

duration T_s and the duration T of the trial. Moreover, the plot in the case that X^+ is Gaussian distributed with equal variances for the **n**- and the **sn**-conditions, i.e. for the standard SDT-assumption, is identical to the different plots when the distribution function of X^+ is given by Eq. (17). So as in the case of the yes–no-experiment, a given value of ψ implies a certain value of d', i.e. the quotient of $\bar{x}^{\mathbf{sn}} - \bar{x}^{\mathbf{n}}$ and $\sqrt{(\sigma_{\mathbf{sn}}^2 + \sigma_{\mathbf{n}}^2)/2}$ is invariant with respect to λ_2 and the other stimulus parameters. It follows that a plot of the probability of a correct response versus the corresponding value of d' does not allow to decide whether detection is by TPS or not, and in particular does not provide any information concerning the value of λ_2.

2.6. Parameter identification

A question not touched so far is whether the values of the free parameters S and λ_2 can be estimated from detection data. It is clear from Eq. (21) that if the value of $\psi(0)$, the probability of a false alarm, is known, then the values of λ_2 and S can no longer be chosen independent of each other, so the number of free parameters is reduced to one. Furthermore, the number of free parameters in the expression for the deterministic function g should be as small as possible. Thus stimuli based on sinusoidal flicker appear to be interesting candidates: if the channel under investigation is linear to a fair degree of approximation, $g(t) = c_0|H(\omega)|\sin(\omega t + \phi(\omega))$. If the stimulus duration is chosen to be $T_s = k/\omega$ with $k = 1, 2, \ldots, n$ and the corresponding threshold contrasts c_{k0} are determined,

and if $\psi(0)$ and the slopes of the psychometric functions for each value of k are known it can be shown that $|H(\omega)|$, S and λ_2 can be determined (Mortensen & Suhl, 1991).

3. Summary and discussion

The main results may be summarised as follows:

1. *TPS and TPD*: Temporal probability summation and temporal peak detection are different processes of detection: if detection is by TPS, there is no explanation of the data in terms of a TPD-model, and if detection is by TPD, there is no TPS-model with respect to which TPD could be conceived as a special case. With respect to the expression (17) for the probability of detection in case of TPS, this means that there is no value of λ_2 which would allow for an approximation in terms of a TPD-model.

   This does not mean that at best one of these modes can be the true mode of detection: it is conceivable that a subject chooses one of the modes depending on the instruction or other conditions of the experiment.

   This result has implications for modelling the read-out or interpretation of the activity of the detecting channel: while in case of detection by TPS a single supra-threshold peak of detection is sufficient to trigger the presence of the stimulus, in case of detection by TPD the maximum of g is extracted from the activity. Possibly the subject takes resort to the TPD-mode if the subject

has to or wants to couple the detection response with the identification of the stimulus. So far, it is unknown how the actual, i.e. conscious perception of a stimulus has to be modelled on the basis of the neural activity of a channel.

2. *Determining the hazard-function*: If detection by TPS is to be modelled, an appropriate hazard-function ϕ has to be found. The definition of ϕ depends on the assumptions concerning the structure of the noise.

   If white, i.e. δ-correlated noise is assumed, the false alarm rates turn out to be independent of the duration T of a trial. This is inconsistent with the notion of TPS.

3. *Additive versus multiplicative noise*: Research on single neurons suggests that the noise observed in the activity of single neurons is multiplicative. However, as pointed out in Comment 1 to the assumptions made in Theorem 2.2, one may assume that detection is the result of a modulation of the activity of cell assemblies (Harris, 2005), where the postulate of stationary noise represents a reasonable approximation, reflecting essential features of TPS.

4. *Effect of the noise*: Given a value of the probability of a false alarm, $p(fa)$, the steepness of a psychometric function for detection by TPS is—for a given stimulus—positively related to the value of λ_2: one may say that the larger the value of λ_2, the greater the sensitivity. However, for given value of $p(fa)$, the value of λ_2 determines the value of S (and vice versa), and an increase of the value of λ_2 implies a corresponding increase of the value of S. So if the steepness of a psychometric function is to be interpreted with respect to the notion of sensitivity as being inversely related to the effect of the noise, one has to say that the effect of the noise depends on λ_2 and S, and the specific type of dependency depends on the value of $p(fa)$.

   In this context it is of interest to consider the relation between λ_2, d', and the probability of detection for a given value of a false alarm $p(fa)$, as discussed in Section 2.5. In Fig. 4(a), the relation between d' and $\lambda_2 = 10^n$ is illustrated for given value of $p(fa)$, the probability and a false alarm, and probability of detection equal to .5; the value of d' increases with that of λ_2. This means that the difference of the means grows faster than the (pooled) variance of the decision variable. (b) shows the corresponding reduction of the stimulus amplitude with increasing values of λ_2.

5. *The relation between d' and $\psi(c)$*: Often one is interested in the value of d' for a given probability of detection, for instance for the value $\psi(c_0) = .5$, or $\psi(c_1) = .75$, where c_0 and c_1 are the corresponding stimulus amplitudes. The standard procedure of computing the value of d' is based on the assumption that the noise (**n**) and the stimulus-plus-noise (**sn**) distributions are Gaussian, with equal variances. If detection is by TPS, the decision variable is neither Gaussian, nor are the variances for the **n** and the **sn** conditions equal. Thus the standard estimate of d' may be incorrect. As may be seen from

Fig. 5(c) and (d), the relation between d' and the probability of detection is invariant with respect to the value of λ_2 for a given value of the probability of a false alarm. However, for $p(fa) = .05$ the assumption of the Gaussian, equal variance case implies an underestimation of d' for values of the probability of detection between .2 and .8, while for $p(fa) = .1$ the value of d' is overestimated for probabilities of detection between .1 and .9. The error is, however, small and will be about zero for $p(fa) \approx .075$.

Fig. 7(c) and (d) show that estimates of d' from the Gaussian-equal-variance case are identical to those resulting from the case of detection by TPS, irrespective of the value of λ_2. This means that if one is interested in estimates of the d'-value corresponding to a particular probability of detection, the hypothesis of TPS and a corresponding estimate of the λ_2-value is not necessary, at least if the assumption of stationary, Gaussian noise is correct.

So when it comes to estimate d' from detection data, the question whether detection is by TPS or not appears to be irrelevant, when a 2AFC-design was employed, and almost irrelevant, when the data are from a yes–no-design with a probability of false alarm in the neighbourhood of .075. The difference between TPS- and TPD-models of detection becomes relevant when the mean internal response $g(t)$ is to be estimated from the data. The fact that TPD is not a special case of TPS, will be illustrated in a separate paper. There, a model of detection by a population of neurons will be presented that provides a natural interpretation of the random variable η, introduced in (5); the model explains why detection by TPD does not predict psychometric functions to be step functions.

Expressions for detection in correlated noise do not transform into expressions based on the assumption of white noise as a limiting case. The fact that the white noise assumption implies inconsistencies will certainly carry over to models of detection by spatial probability summation, or probability summation among different neural channels. For instance, Bonneh and Sagi (1998) and Usher, Bonneh, Sagi, and Herrmann (1999) report data that are not compatible with independent channels for certain stimulus elements, e.g. Gabor patches. In fact, Usher et al.'s attempt to interpret their data employing a network model. Such models allow in principle to cope with correlations among spatially separated channels whose activity is represented by non-white stochastic processes. Progress is still possible in the quest for a generally-applicable spatial detection model (du Buf, 2005). This type of modelling is more demanding, but certainly also more rewarding than fitting Quick's formula.

Acknowledgments

pointing out to me the relevant results (cf. Leadbetter, Lindgren, & Rootzén, 1983, Chapter 8) required to derive Theorem 2.2. I am equally indebted to Prof. Hans du Buf, University of the Algarve, for his critical comments and the constructive discussions of a previous draft of this paper.

Appendix A

A.1. δ-correlated noise

A.1.1. Proof of Eq. (12)

Suppose the random variables $\xi_i = \xi(t_i)$ are Weibull-distributed, so that

$$P(\xi_i \leqslant x) = \exp(-(\eta_0 - x)^\beta), \quad x \leqslant \eta_0, \quad \beta > 0, \tag{30}$$

(Johnson, Kotz, & Balakrishnan, 1994). Since $P(X(t) \leqslant S) = P(\xi(t) + g(t) \leqslant S) = P(\xi(t) \leqslant S - g(t))$, we can define $F_\xi(S - g(t_i)) = P(\xi_i \leqslant S - g(t_i))$. Consider the subdivision of $J = [0, T]$ into n intervals J_i as introduced in Section 2.2, and let $X_i^* = \max_{t \in J_i} X(t) = g(t_i^*) + \xi_i^*$ (see (9)). The distribution function of the X_i^*, i.e. of the ξ_i^*, is not given by (30), but will approach (30) for increasing value of n, i.e. $\xi_i^* \to \xi_i$ for $n \to \infty$. So, for sufficiently large value of n one may replace the ξ_i^* by the ξ_i and since δ-correlated noise is assumed one may write

$$P_D(n) \approx 1 - \prod_{i=1}^n F_\xi(S - g_i)$$

$$= 1 - \exp\left(-\sum_{i=1}^n (\eta_0 - (S - g_i))^\beta\right), \tag{31}$$

where g_i is the value of $g(t)$ at some time $t_i \in J_i$; the $\approx$-sign appears because for finite n the ξ_i^* will not be identical to the ξ_i. To consider the case of continuous t one would like to pass to the limit $n \to \infty$, implying $\Delta t \to 0$. Let $x_i = S - g_i$. As is well known from extreme-value theory, $n \to \infty$ requires a re-scaling of the x_i such that the product in Eq. (15) does not degenerate and approach 0. The re-scaling means that so-called norming constants a_n and b_n have to be found such that $F_\xi(b_n x_i + a_n) \to 1$ for $n \to \infty$, for all i, and $0 < \prod_i F_\xi(b_n x_i + a_n) < 1$. For the Weibull distribution, the limiting distribution is of the form $\exp(-(-x)^\beta)$, with $a_n = \sup\{x | F(x) < 1\} = \eta_0$ and $b_n = a_n - \inf\{x | 1 - F(x) \leqslant 1/n\}$ (Galambos, 1978, p. 52). Now $\sup\{x | F(x) < 1\} = \eta_0$, and $\inf\{x | 1 - F(x) \leqslant 1/n\} = \eta_0 - (-\log(1 - 1/n))^{1/\beta}$. For sufficiently large values of n one has $\log(1 - 1/n) \approx -1/n$, so $b_n \to (1/n)^{1/\beta}$ and $b_n x_i + a_n \to (1/n)^{1/\beta} x_i + \eta_0$, and from Eq. (31) it follows that

$$\sum_i (\eta_0 - (1/n)^{1/\beta} x_i - \eta_0)^\beta = \sum_i (1/n)(g_i - S)^\beta$$

$$= \frac{1}{T} \sum_i (g_i - S)^\beta \frac{T}{n}, \tag{32}$$

and for $n \to \infty$ one has $T/n \to dt$ and $P_D(n) \to \psi(c)$, and one finds

$$\psi(c) = 1 - \exp\left(-\frac{1}{T} \int_0^T (g(t) - S; c)^\beta \, dt\right),$$

$$0 \leqslant g(t; c) - S, \quad \forall t. \tag{33}$$

For $g(t) = 0$ for all t the right-hand side yields the limiting form $\exp[-(-x)^\beta]$ mentioned above, with $x = S \leqslant 0$. Since $\xi(t) = S - g(t)$, the requirement $0 \leqslant g(t; c) - S$ implies $g(t) - S = -\xi(t) \geqslant 0$, i.e. $\xi(t) \leqslant 0$. $\square$

A.1.2. Proof of Eq. (13)

The approach is identical to the one in Section A.1.1 except for the assumption that the ξ_i are independent Gaussian random variables, with variance $\sigma^2 = 1$ and mean zero, $E(\xi_i) = 0$ for all i. The assumption $\sigma^2 = 1$ represents no restriction of generality; it just means that the values of the mean or deterministic response g are expressed in units of σ^2, which may of course assume any value. As in Section A.1.1, one has to apply suitable norming constants a_n and b_n such that $F_\xi(b_n(S - g_i) + a_n) \to 1$ for $n \to \infty$. The norming constants are known to be

$$b_n = (2 \log n)^{-1/2}, \tag{34}$$

$$a_n = \sqrt{2 \log n} - \frac{\log \log n + \log(4\pi)}{2\sqrt{2 \log n}} \tag{35}$$

(Leadbetter et al. (1983, p. 14); for a more recent reference, see also Beirlant, Goegebeur, Segers, & Teugels, 2004). Generally,

$$\prod_i P(\xi \leqslant x_i) = \exp\left(\sum_i \log P(\xi \leqslant x_i)\right), \quad x_i = S - g_i. \tag{36}$$

Now $P(b_n x_i + a_n) \to 1$, and $\log P \approx -(1 - P)$ for $P \to 1$, so one has

$$\prod_i P(\xi \leqslant x_i) \approx \exp\left(-\sum_{i=1}^n (1 - P)(\xi \leqslant b_n x_i + a_n)\right) \tag{37}$$

$$= \exp\left(-\sum_{i=1}^n P(\xi > b_n x_i + a_n)\right). \tag{38}$$

For large x the approximation

$$P(\xi > x) \approx \frac{1}{x\sqrt{2\pi}} \exp(-x^2/2) \tag{39}$$

holds. Substituting $b_n x_i + a_n$ for x gives

$$P(\xi > b_n x_i + a_n) \approx \frac{1}{(b_n x_i + a_n)\sqrt{2\pi}} \exp(-(b_n x_i + a_n)^2/2). \tag{40}$$

Writing

$$\tfrac{1}{2}(b_n x_i + a_n)^2 = \tfrac{1}{2}(b_n^2 x_i^2 + a_n^2 + 2x_i b_n a_n),$$

and since $b_n \to 0$ for $n \to \infty$, it follows that $b_n^2 x_i^2 \to 0$.

Now

$$\frac{\log \log n + \log(4\pi)}{2\sqrt{2 \log n}} \to 0, \quad n \to \infty,$$

(see Eq. (35)) and it further follows that $a_n^2 \to 2 \log n$. Also, by combining Eqs. (34) and (35),

$$a_n b_n = 1 - \frac{\log \log n + \log(4\pi)}{4 \log n} \to 1, \quad n \to \infty,$$

and

$$\exp(-\tfrac{1}{2}(b_n^2 x_i^2 + a_n^2 + 2x_i b_n a_n))$$
$$= \exp(-(b_n x_i)^2/2) \exp(-a_n^2/2) \exp(-x_i a_n b_n),$$

with $\exp(-(b_n x_i)^2/2) \to 1$, $\exp(-a_n^2/2) \to 1/n$, $\exp(x a_n b_n) \to \exp(-x)$. Therefore, $\exp(-(b_n x_i + a_n)^2/2) \to -\exp(-x_i)/n$. Substituting $S - g(t_i)$ for x_i one has from Eq. (37)

$$\prod_i P(\xi \leqslant x_i) \to \exp\left(-\frac{1}{n}\sum_{i=1}^n \exp[-(S - g(t_i))]\right). \tag{41}$$

With

$$\frac{1}{n}\sum_{i=1}^n \exp[-(S - g(t_i))] = \frac{1}{T}\sum_{i=1}^n \exp[-(S - g(t_i))]\frac{T}{n}$$
$$\to \frac{1}{T}\int_0^T e^{-(S-g(t))}\,dt, \quad n \to \infty, \tag{42}$$

one arrives at the psychometric function

$$\psi_{\mathrm{G}}(c) = 1 - \exp\left(-\frac{1}{T}\int_0^T e^{-(S-g(t))}\,dt\right). \quad \Box \tag{43}$$

A.2. Gaussian non-white noise

Let $\xi_i^* = \max_{t \in J_i} \xi(t)$. The distribution of ξ_i^* depends on the width Δt of the J_i, and it will be useful to make this dependency explicit by writing $\xi_i^*(\Delta t)$. For sufficiently large values of Δt, where the term "sufficiently large" has to be defined with respect to the autocorrelation function $R(\tau)$ of ξ_t, the $\xi_i^*(\Delta t)$ are approximately stochastically independent (Leadbetter et al., 1983), so that one may write

$$P(X^+ \leqslant S) \approx \prod_{i=1}^n P[\xi_i^*(\Delta t) \leqslant S - g(t)], \quad \forall t \in J_i. \tag{44}$$

In order to arrive at an expression for $P[\xi_i^*(\Delta t) \leqslant S - g(t), \forall t \in J_i]$ in the above equation, the following notions have to be introduced. Let z be a random variable with density given by $f(z) = \exp(-z^2/2)/\sqrt{2\pi}$, and let

$$\Phi(u) = \int_{-\infty}^u f(z)\,dz, \quad \mu(u) = \frac{\sqrt{\lambda_2}}{2\pi}\exp(-u^2/2) = E(N_u(1)), \tag{45}$$

where $\mu(u)$ is known as the *Rice formula*, and $E(N_u(1))$ indicates that $\mu(u)$ gives the expected number of up-

crossings per unit time (Leadbetter et al., 1983, p. 153). As shown by Leadbetter et al. (1983, Lemma 8.2.1, p. 166), the following statements hold:

1. For all $\Delta t > 0$, $P(\xi_i^*(\Delta t) > u) \leqslant 1 - \Phi(u) + \mu(u)\Delta t$,
2. Given $\Theta < 1$ there exists $\Delta t_0 = \Delta t_0(\Theta)$, such that for $0 \leqslant \Delta t \leqslant \Delta t_0$,
$$P(\xi_i^*(\Delta t) > u) \geqslant 1 - \Phi(u) + \Theta\Delta t.$$

The inequalities can be summarised into a single inequality

$$1 - \Phi(u) + \Theta\mu(u)\Delta t \leqslant P(\xi_i^*(\Delta t) > u) \leqslant 1 - \Phi(u) + \mu(u)\Delta t, \tag{46}$$

where $u = S - g(t)$. Since Θ can be chosen arbitrarily close to 1, Eq. (46) implies

$$P(\xi_i^*(\Delta t) > u) \to 1 - \Phi(u) + \mu(u)\Delta t \quad \text{for } \Theta \to 1.$$

After replacing u by $S - g(\tau_i)$, with $\tau_i \in J_i$, Eq. (44) can be written in the form

$$P(X^+ \leqslant S) \approx \prod_{i=1}^n [1 - (1 - \Phi(S - g(\tau_i)) + \mu(S - g(\tau_i))\Delta t]$$
$$= \prod_{i=1}^n [\Phi(S - g(\tau_i)) - \mu(S - g(\tau_i))\Delta t].$$

For $S \to \infty$ one has $\Phi(S - g(\tau_i)) \to 1$, and one finds

$$P(X^+ \leqslant S) \approx \prod_{i=1}^n [1 - \mu(S - g(\tau_i))\Delta t]$$
$$= \exp\left[\sum_{i=1}^n \log(1 - \mu(S - g(\tau_i))\Delta t)\right].$$

Since for large values of n the value of Δt and thus that of the term $\mu(S - g(\tau_i))\Delta t$ will become small compared to 1, one can make use of the fact that $\log(1 - \varepsilon) \approx -\varepsilon$, $\varepsilon \ll 1$, so that

$$P(X^+ \leqslant S) \approx \exp\left[-\sum_{i=1}^n \mu(S - g(\tau_i))\Delta t\right].$$

Making use of Eq. (45) one finds

$$\exp\left[-\sum_{i=1}^n \mu(S - g(\tau_i))\Delta t\right]$$
$$= \exp\left[-\frac{\sqrt{\lambda_2}}{2\pi}\sum_{i=1}^n \exp\left(-\frac{(S - g(\tau_i))^2}{2}\right)\Delta t\right].$$

Since $\Delta t = T/n$ one has

$$\lim_{n \to \infty} \sum_{i=1}^n \exp\left(-\frac{(S - g(\tau_i))^2}{2}\right)\frac{T}{n}$$
$$= \int_0^T \exp\left(-\frac{(S - g(t))^2}{2}\right)dt,$$

so that, with $\psi(c) = 1 - P(X^+ \leqslant S|c)$, it follows that finally

$$\psi(c) = 1 - \exp\left[-\frac{\sqrt{\lambda_2}}{2\pi}\int_0^T \exp\left(-\frac{(S-g(t))^2}{2}\right)dt\right]. \qquad \square$$

(47)

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